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Relaxed Trajectories of Integrodifferential Equations and Optimal Control on Banach Space

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Abstract—Relaxation optimal control for a class of integrodifferential equation associated with fractional power operator is investigated. By Gronwall lemma with singularity and delay the existence of mild solutions of integrodifferential equations is proved. Introducing finitely additive measure control, we convexificate original control systems and obtain corresponding relaxed control systems. Approximation results and equivalent relationship between control differential equations and integrodifferential inclusion are given. Existence of relaxed optimal control is verified and one relaxed optimal control is just an original one under generalized Cesari conditions.

Keywords—Integrodifferential equation, Semigroup, Compactness, Optimal control, Relaxation.

1. INTRODUCTION

It is well known that to guarantee existence of an optimal "state-control" pair we need a convexity hypothesis on a certain orientor field. When this convexity hypothesis is no longer satisfied, to have optimal solutions we need to pass to a larger system, in which the orientor field has been convexified and measure-valued controls, so-called relaxed control, have been introduced.

For finite-dimensional control systems, this problem has already been studied in the literature (see [1]). For infinite-dimensional control systems, Ahmed dealt with this problem and introduced measure-valued control in which control space is compact and values of relaxed controls are countable additive measures (see [2]). Then Papageorgiou and other authors including us continue to discuss this problem (see [3–5]). Since 1991, Fattorini has been working with relaxed controls whose values are finitely additive measures. By his approach we can cope with such a control set which is a normal topology space even arbitrary set (see [6,7]). However to our knowledge, there are few results on relaxation of integrodifferential equations, particularly integrodifferential inclusions.

In this paper we study the relaxation for systems governed by integrodifferential equations associated with fractional power operator and optimal control. Based on discussing integrodifferential

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equations, we introduce relaxed controls whose values are finitely additive measures and overcome the difficulties brought by the integral operator. We discuss a series of relative problems. At first some approximation results are given. Then introducing integrodifferential inclusion and giving the definition of solutions of integrodifferential inclusion we prove the equivalent relationship between control systems and integrodifferential inclusion. Using weak compactness and denseness of relaxed controls, we obtain existence of relaxed optimal controls. Under generalized Cesari condition introduced in this paper for integrodifferential equations, we assert the relaxed optimal control is just an original optimal control.

The paper is organized as follows. In Section 2, using Gronwall lemma with singularity and delay we discuss the integrodifferential equations. The original control systems and corresponding relaxed control systems are given in Section 3. Section 4 contributes to relaxation theorem, and we present some approximation results. Integrodifferential inclusion is discussed in Section 5. Existence of relaxed optimal controls and original optimal controls is proved in the last section.

2. INTEGRODIFFERENTIAL EQUATIONS

In order to discuss integrodifferential equations, we need the following Gronwall lemma with singularity and delay.

Let E be a Banach space, $I = [0,T]$, $I_r = [-r,T]$, and $C(I_r;E)$ be the space of all continuous functions from I_r to E with supernorm. $C(I_r;E)$ is a Banach space. For $x \in C(I_r;E)$, we set

$$x_t(\tau) = x(t+\tau), \quad -r \leq \tau \leq 0, \quad \|x_t\|_B = \sup_{0\leq\tau\leq t} \|x(\tau)\|, \qquad \text{for every } t > 0.$$

LEMMA 2.1. *(See [8, Lemma 1.2].) Let $0 < \alpha < 1$ and $q > 1/(1-\alpha)$. Suppose $x \in C(I_r;E)$ satisfies the following inequality:*

$$\|x(t)\| \leq \omega(t) + \int_0^t (t-\tau)^{-\alpha} l(\tau)\|x(\tau)\|\, d\tau + \int_0^t (t-\tau)^{-\alpha}\|x_\tau\|_B\, d\tau,$$

where $\omega \in C(I_r;\bar{R}^+)$ is increasing and $l \in L^q(I,\bar{R}^+)$. Then there exists a constant $M \geq 0$ such that

$$\|x(t)\| \leq M\omega(t), \qquad t \in I_r.$$

Now we consider the following integrodifferential equation which has applications in viscoelastic problems:

$$\begin{aligned} \dot{y}(t) &= Ay(t) + g(t,y(t)) + \int_{-r}^t \theta(t-s)h(s,y(s))\, ds, && t \in (0,T], \\ y(t) &= \varphi(t), && t \in [-r,0], \end{aligned} \tag{2.1}$$

where initial value $\varphi \in C([-r,0],E_\alpha)$.

We introduce fractional power operator A^α and fractional power spaces $(E_\alpha, \|\cdot\|_\alpha)$ $(0 \leq \alpha \leq 1)$ in the usual way (see [9,10]).

The mild solution of equation (2.1) is understood as the solution of following integral equation:

$$\begin{aligned} y(t) &= T(t)\varphi(0) + \int_0^t T(t-\tau)g(\tau,y(\tau))\, d\tau \\ &\quad + \int_0^t T(t-\tau)\left[\int_{-r}^\tau \theta(\tau-s)h(s,y(s))\, ds\right] d\tau, && t \in (0,T], \\ y(t) &= \varphi(t), && t \in [-r,0]. \end{aligned} \tag{2.2}$$

Define $\omega(t) = T(t)\varphi(0) + \int_0^t T(t-\tau)[\int_{-r}^0 \theta(\tau - s)h(s,\varphi(s))\,ds]\,d\tau$, then $\omega(\cdot) \in C(I, E_\alpha)$, $\omega(0) = \varphi(0)$. We can consider the following integral equation (2.3) instead of (2.2):

$$\begin{aligned} y(t) &= \omega(t) + \int_0^t T(t-\tau)g(\tau, y(\tau))\,d\tau \\ &\quad + \int_0^t T(t-\tau)\left[\int_0^\tau \theta(\tau - s)h(s, y(s))\,ds\right] d\tau, \qquad t \in (0, T], \\ y(t) &= \varphi(t), \qquad t \in [-r, 0]. \end{aligned} \tag{2.3}$$

We impose the following assumptions.

ASSUMPTION [A]. *$-A$ is the infinitesimal generator of an analytic semigroup $\{T(t),\, t \geq 0\}$ in E and $0 \in \rho(A)$.*

ASSUMPTION [G]. *$g : I \times E_\alpha \longrightarrow E$ satisfies*

(1) *$t \longmapsto g(t, y)$ is measurable, and there exists $l_1(\cdot, e) \in L^\infty(I_r, \bar{R}^+)$ such that*

$$\|g(t, y_1) - g(t, y_2)\| \leq l_1(t, e)\|y_1 - y_2\|_\alpha, \qquad \text{for every } e > 0,$$

for $t \in I$ a.e. and provided $\|y\|_\alpha, \|y_1\|_\alpha, \|y_2\|_\alpha \leq e$.

(2) *There exists $C_1(\cdot) \in L^q(I, \bar{R}^+)$ $(q > 1/(1-\alpha))$ such that*

$$\|g(t, y)\| \leq C_1(t)(1 + \|y\|_\alpha), \qquad t \in I, \quad y \in E_\alpha.$$

ASSUMPTION [H]. *Let $\gamma_1 > 1$, and $1/\gamma_1 + 1/\gamma_2 = 1$. $\theta \in L^{r_1}[0, T+r]$. $h : I_r \times E_\alpha \longrightarrow E$ satisfies*

(1) *$t \longmapsto h(t, y)$ is measurable, and there exists $l_2(\cdot, e) \in L^\infty(I_r, \bar{R}^+)$ such that*

$$\|h(t, y_1) - h(t, y_2)\| \leq l_2(t, e)\|y_1 - y_2\|_\alpha, \qquad \text{for every } e > 0,$$

for $t \in I_r$ a.e. and provided $\|y_1\|_\alpha, \|y_2\|_\alpha \leq e$.

(2) *There exists $C_2(\cdot) \in L^{r_2}(I, \bar{R}^+)$ such that*

$$\|h(t, y)\| \leq C_2(t)(1 + \|y\|_\alpha), \qquad t \in I, \quad y \in E_\alpha.$$

By standard arguments and Gronwall lemma with singularity and delay, one can verify the following main result on integrodifferential equations.

THEOREM 2.A. *Under Assumptions [A], [G], and [H], integrodifferential equation (2.3) has a unique solution $y \in C(I_r, E_\alpha)$ for every $\varphi \in C([-r, 0], E_\alpha)$. Further, there exists M depending on Φ such that*

$$\|y_1 - y_2\|_{C(I_r, E_\alpha)} \leq M\|\varphi_1 - \varphi_2\|_{C([-r,0], E_\alpha)},$$

where $\varphi_1, \varphi_2 \in \Phi$ which is a bounded set of $C([-r, 0], E_\alpha)$, y_1 (respectively, y_2) is the solution of (2.3) corresponding to φ_1 (respectively, φ_2).

3. ORIGINAL CONTROL SYSTEMS AND RELAXED CONTROL SYSTEMS

Suppose the following.

ASSUMPTION [U]. *U is a normal topology space. U_{ad} consists of all U-valued strongly measurable functions.*

ASSUMPTION [G$_1$]. *$g : I \times E_\alpha \times U \longrightarrow E$ satisfies*

(1) *$t \mapsto g(t, y, u)$ is measurable; $u \mapsto g(t, y, u)$ is locally uniformly continuous.*

(2) *For every $e > 0$, there exists $l_3(\cdot, e) \in L^\infty(I, \bar{R}^+)$ such that*

$$\|g(t, y_1, u) - g(t, y_2, u)\|_E \leq l_3(t, e)\|y_1 - y_2\|_\alpha$$

provided $t \in I$, $u \in U$, $\|y\|_\alpha, \|y_1\|_\alpha, \|y_2\|_\alpha \leq e$.

(3) *There exists $C_3(\cdot) \in L^q(I, \bar{R}^+)$ $(q > 1/(1-\alpha))$ such that*

$$\|g(t, y, u)\| \leq C_3(t)\left(1 + \|y\|_\alpha\right), \qquad t \in I, \quad y \in E_\alpha.$$

ASSUMPTION $[\mathrm{H}_1]$. *$\theta \in L^{r_1}(0, T+r)$. $h : I_r \times E_\alpha \times U \longrightarrow E$ satisfies*

(1) *$t \mapsto h(t, y, u)$ is measurable; $u \mapsto h(t, y, u)$ is locally uniformly continuous.*

(2) *For every $e > 0$ there exists $l_4(\cdot, e) \in L^\infty(I_r, \bar{R}^+)$ such that*

$$\|h(t, y_1, u) - h(t, y_2, u)\|_E \leq l_4(t, e)\|y_1 - y_2\|_\alpha$$

provided $t \in I_r$, $u \in U$, $\|y_1\|_\alpha, \|y_2\|_\alpha \leq e$.

(3) *There exists $C_4(\cdot) \in L^{r_2}(I_r, \bar{R}^+)$ such that*

$$\|h(t, y, u)\| \leq C_4(t)(1 + \|y\|_\alpha), \qquad t \in I_r, \quad y \in E_\alpha, \quad u \in U.$$

We consider the following original control systems:

$$\begin{aligned} \dot{y}(t) &= Ay(t) + g(t, y(t), u(t)) \\ &\quad + \int_{-r}^{t} \theta(t-s)h(s, y(s), u(s))\, ds, && t \in (0, T], \\ y(t) &= \varphi(t), && t \in [-r, 0], \quad u \in U_{ad}. \end{aligned} \tag{3.1}$$

By Theorem 2.A, one can easily get the following existence theorem.

THEOREM 3.A. *Under Assumptions [A], $[\mathrm{G}_1]$, and $[\mathrm{H}_1]$, original control system (3.1) has a unique mild solution on interval I_r.*

In order to introduce relaxed control system corresponding to (3.1), we need to give some explanations (see [6, Chapter 12]). We denote by $\mathrm{BC}(U)$ the space of all bounded continuous functions defined in U. Endowed with the supremum norm, $\mathrm{BC}(U)$ is a Banach space. For simplicity, we denote $\mathrm{BC}(U)$ by X. Let $\Phi(C)$ be a field generalized by collection C of all closed sets of U and $\sum_{\mathrm{rba}}(U)$ the space of all regular bounded finitely additive measures on measurable space $(U, \Phi(C))$. For $\mu \in \sum_{\mathrm{rba}}(U)$, $|\mu|$ denotes the total variation of μ.

LEMMA 3.1. *(See [6, Theorem 12.4.6].) $X^* = \mathrm{BC}(U)^*$ can be identified algebraically and metrically with $\sum_{\mathrm{rba}}(U)$ with the norm $\|\mu\|_{\sum_{\mathrm{rba}}(U)} = |\mu|(U)$. The duality pairing of $X = \mathrm{BC}(U)$ and $X^* = \sum_{\mathrm{rba}}(U)$ is given by*

$$\langle f, \mu \rangle = \int_U f(\sigma)\mu\,(d\sigma), \qquad \text{for } f \in X, \quad \mu \in X^*.$$

Suppose $L^1(I_r, X)$ is the space of all (equivalence class of) strongly measurable X-valued functions $f(\cdot)$ defined on I_r such that

$$\|f\|_1 = \int_{I_r} \|f(t)\|_X\, dt < +\infty.$$

$L^1(I_r, X)$ is a Banach space. We denote by $L_w^\infty(I_r, X^*)$ is the space of all X^*-valued X-weakly measurable functions $g(\cdot)$ such that there exists $C > 0$ with

$$|\langle g(t), y\rangle| \leq C\|y\|_X, \qquad \text{a.e. } t \in I_r, \quad \text{for each } y \in X,$$

where $C \geq 0$ independent of y, null set fails to hold may depend on y (see [6, Section 12.2]).

LEMMA 3.2. *(See [6, Theorem 12.2.11].) $L^1(I_r, X)^*$ is isometrically isomorphic to $L_w^\infty(I_r, X^*)$. The duality pairing between both spaces given by*

$$\langle\!\langle g, f \rangle\!\rangle = \int_{I_r} \langle g(t), f(t)\rangle\, dt.$$

Definition 3.1. *The space $R(I_r, U)$ of relaxed controls consists of all $\mu(\cdot)$ in $L_w^\infty(I_r, X^*) = L^1(I_r, X)^*$ such that:*

(a) *let $f \in L^1(I_r, X)$ be such that $f(t,\sigma) \geq 0$ for $\sigma \in U$ a.e. on $t \in I_r$, then*

$$\int_{I_r} \int_U f(t,\sigma)\mu_t(d\sigma)\,dt \geq 0.$$

(b) *Let $\chi(\cdot)$ be the characteristic function of a measurable set $e \subseteq I_r$, and $1 \in \mathrm{BC}(U)$ be the function $1(\sigma) = 1$ then*

$$\int_{I_r} \int_U (\chi(t) \otimes 1(\sigma))\mu_t(d\sigma)\,dt = |e|.$$

It follows from Lemma 12.5.1 of [6] that for $\mu \in R(I_r, U)$, we have $\|\mu\|_{L_w^\infty(I_r,X^*)} \leq 1$. Furthermore for $\mu \in R(I_r, U)$ there exists ν in equivalence class of μ in $L_w^\infty(I_r, X^*)$ such that

$$\nu(t) \geq 0, \quad \nu(t, U) = 1, \qquad \text{a.e. } t \in I_r.$$

In particular

$$\|\nu(t)\| = 1, \qquad -r \leq t \leq T.$$

(See [6, Theorem 12.5.8].)

The following compactness result is cruel.

Lemma 3.3. *(See [6, Theorem 12.5.9].) Let $\{\mu_k\}$ be a sequence or generalized sequence in $R(I_r, U)$. Then there exists a generalized subsequence $L^1(I_r, X)$-weakly convergent in $L_w^\infty(I_r, X^*)$ to $\mu(\cdot) \in R(I_r, U)$.*

Sometimes, using another equivalent definition of $R(I_r, U)$ is more convenient. $\prod_{\mathrm{rba}}$ denotes the set of all probability measures μ in $\sum_{\mathrm{rba}}(U)$. Dirac measure with mass at u is denoted by the functional notation is $\delta(\cdot - u)$ or by δ_u. Then set $D = \{\delta_u,\ u \in U\}$ of all Dirac measures is a subset of $\prod_{\mathrm{rba}}$.

Lemma 3.4. *(See [6, Lemma 12.6.4].) $\prod_{\mathrm{rba}}$ is $\mathrm{BC}(U)$-weakly compact in $\sum_{\mathrm{rba}}(U)$, also $\mathrm{BC}(U)$-weakly closed. Let $\overline{\mathrm{conv}}$ denote closed convex hull (closure taken in the weak $\mathrm{BC}(U)$-topology). Then*

$$\prod_{\mathrm{rba}} = \overline{\mathrm{conv}}(D).$$

Elements of D are extrema points of $\prod_{\mathrm{rba}}$. If v, u belong to the same equivalence class, we denote by $v \approx u$. If X is separable, the equivalent relation in $L_w^\infty(I_r, X^*)$ is equal almost everywhere. We have

$$R(I_r, U) = \left\{ u \in L_w^\infty(I_r, X^*),\ \exists\, v \text{ s.t. } v \approx u \text{ and } v(t) \in \prod_{\mathrm{rba}},\ t \in I_r \right\}.$$

$PC(I_r; \prod_{\mathrm{rba}})$ is the set of all $\prod_{\mathrm{rba}}$-valued functions defined in I_r and constant in intervals $t_{j-1} \leq t < t_j$ where $-r = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$. Space $PC(I_r; D)$ is similarly defined but the functions are D-valued.

Lemma 3.5. *(See [6, Theorem 12.6.7].) $PC(I_r; D)$ is $L^1(I_r; X)$ weakly dense in $R(I_r, U)$. If X is separable, $PC(I_r; D)$ is $L^1(I_r; X)$-weakly sequentially dense in $R(I_r, U)$.*

If $u(\cdot) \in U_{ad}$, let δ be the Dirac delta, set $u(t) = \delta(\cdot - u(t))$, $u(\cdot)$ is strongly measurable, then $u(\cdot) \in R(I_r, U)$. Define $G(t,y)(H(t,y))$ as

$$\langle y^*, G(t,y)\mu \rangle = \int_U \langle y^*, g(t,y,\sigma) \rangle\, \mu(d\sigma), \qquad \forall\, y^* \in E^*, \quad \mu \in X^*, \tag{3.2}$$

$$\langle y^*, H(t,y)\mu \rangle = \int_U \left\langle y^*, \int_{-r}^t \theta(t-s) h(s, y(s)\sigma)\, ds \right\rangle \mu(d\sigma), \qquad \forall\, y^* \in E^*, \quad \mu \in X^*. \tag{3.2*}$$

LEMMA 3.6. *Let E be reflexive and separable. If Hypothesis [G_1] (respectively, Hypothesis [H_1] holds, (3.2) (respectively, (3.2*)) defines an operator-valued function*

$$(t, y) \longmapsto G(t, y) \text{ (respectively, } H(t, y)) : I \times E_\alpha \to \pounds\left(X^*, E\right), \qquad \text{for every } \mu \in X^*.$$

Further, for $\mu(\cdot) \in R(I_r, U)$, $G(\cdot, \cdot)\mu(\cdot)$ (respectively, $H(\cdot, \cdot)\mu(\cdot)) : I_r \times E_\alpha \to E$ satisfies Assumption [G] (respectively, [H]).

PROOF. For fixed $t \in I$, $y \in E_\alpha$, by Assumption [G_1], $g(t, y, \cdot) \in BC(U, E)$. For any given $y^* \in E^*$, $\langle y^*, g(t, y, \cdot)\rangle \in \mathrm{BC}(U)$. Hence,

$$\langle y^*, G(t, y)\mu\rangle = \int_U \langle y^*, g(t, y, \sigma)\rangle\, \mu\,(d\sigma)$$

is meaningful and defines a unique element $G(t, y)\mu$ of $E^{**} = E$ for $\mu \in X^*$. $G(t, y) \in \pounds(X^*, E)$. For $\mu(\cdot) \in R(I_r, U)$, by Assumption [G_1], one can verify that

$$\|G(t, y)\mu\|_E \leq C_3(t)\,(1 + \|y\|_\alpha)\, \|\mu\|_{L^\infty_w(I_r, X^*)}$$

and

$$\|G(t, y_1)\mu - G(t, y_2)\mu\| \leq l_3(t, e)\|y_1 - y_2\|_\alpha \|\mu\|_{X^*}$$

provided $\|y_1\|_\alpha, \|y_2\|_\alpha \leq e$. This means $G(\cdot, \cdot)\mu(\cdot)$ satisfies Assumption [G]. Similarly, we obtain the same results on $H(\cdot, \cdot)\mu(\cdot)$ for $\mu(\cdot) \in R(I_r, U)$. ∎

Now, we can consider relaxed system corresponding to (3.1).

$$\begin{aligned} \dot{y}(t) + Ay(t) &= G(t, y(t))\mu(t) + H(t, y(t))\mu(t), && t \in (0, T], \\ y(t) &= \varphi(t), && t \in [-r, 0], \quad \mu \in R\left(I_r, U\right). \end{aligned} \tag{3.3}$$

One can directly obtain the following existence result from Theorem 2.A.

THEOREM 3.B. *Under Assumptions [A], [G_1], and [H_1], relaxed system (3.3) has a unique mild solution for every $\mu(\cdot) \in R(I_r, U)$ on I_r.*

4. RELAXATION THEOREMS

Suppose the following.

ASSUMPTION [A_1]. *$(-A)$ satisfies all Assumption [A] and A has compact resolvent.*

LEMMA 4.1. *If [A_1] holds, $(-A)$ generates compact analytic semigroup $\{T(t),\ t \geq 0\}$. The operator Γ is given by*

$$(\Gamma g)(t) = \int_0^t T(t - \tau) g(\tau)\, d\tau, \qquad \text{for every } g(\cdot) \in L^q(I, E).$$

Then $\Gamma : L^q(I, E) \longrightarrow C(I, E_\alpha)$ is compact provided $0 < \alpha < 1/p$.

PROOF. Obviously $T(t - \cdot)g(\cdot)$ is measurable and

$$\begin{aligned} \|(\Gamma g)(t)\|_\alpha &\leq C \int_0^t (t - \tau)^{-\alpha} \|g(\tau)\|_E\, d\tau \\ &\leq C \left(\int_0^T (t - \tau)^{-\alpha p}\, d\tau\right)^{1/p} \left(\int_0^T \|g(\tau)\|_E^q\, d\tau\right)^{1/q} \\ &< +\infty. \end{aligned}$$

Supposing $g \in Q$, where Q is a bounded set of $L^q(I,E)$, it is easy to see that $\Gamma(Q)$ is bounded in $C(I,E_\alpha)$. Choose β such that $0<\alpha<\beta<1/p$. For $t \in (0,T]$, we have $(\Gamma g)(t) \in E_\beta$. Since E_β is compactly embedded in E_α, it is easy to show that $\{\Gamma g(t)\}$ is precompact in E_α for $t \in I$ and $g \in Q$. For every $g \in Q$, $0 \le t_1 < t_2 \le \bar{t}$, there exists a C^* dependent only on M, and C, such that

$$\left\|\Gamma g\left(t_2\right)-\Gamma g\left(t_1\right)\right\|_\alpha \leq C^*\left(\left|t_2-t_1\right|^{\beta-\alpha}+\left|t_2-t_1\right|^{1-\alpha}\right).$$

This means that $\Gamma(Q)=\{\Gamma g \mid g \in Q\}$ is equicontinuous. Hence $\Gamma(Q)$ is a relatively compact set in $C(I,E_\alpha)$. ∎

Now we can give the following relaxed theorem.

THEOREM 4.A. *Let E be reflexive and separable. Assumptions [A_1], [G_1], and [H_1] hold. Let $y(\cdot,\mu)$ be the solution of (3.3) corresponding to μ, then for every $\varepsilon>0$, there exists $u(\cdot) \in PC(I_r;D)$ such that solution $y(\cdot,u)$ of (3.1) corresponding to u satisfies*

$$\|y(t,\mu)-y(t,u)\|_\alpha \leq \varepsilon, \qquad t \in I_r.$$

PROOF. By Lemma 3.5, $PC(I_r,D)$ is $L^1(I_r,X)$-weakly sequentially dense in $R(I_r,U)$. For $\mu \in R(I_r,U)$ there exists a generalized sequence $\{u_k\} \subset PC(I_r,D)$ such that $u_k(\cdot) \longrightarrow \mu(\cdot) L^1(I_r,X)$-weakly in $L_w^\infty(I_r,X^*)$.

Let $y_k(\cdot)=y(\cdot,u_k)$ be the solution of (3.1) corresponding to u_k and $y(\cdot)=y(\cdot,\mu)$ be the solution of (3.3) corresponding to μ. Define

$$\eta_k(t)=\int_0^t T(t-\tau)\xi_k(\tau)\,d\tau=\left(\Gamma \xi_k\right)(t), \qquad \omega_k(t)=\int_0^t T(t-\tau)\zeta_k(\tau)\,d\tau=\left(\Gamma \zeta_k(t)\right),$$

where

$$\begin{aligned}
\xi_k(t) &= G(t,y(t))\left(u_k(t)-\mu(t)\right), \\
\zeta_k(t) &= \int_{-r}^t \theta(t-s)\left[H(s,y(s))\left(u_k(s)-\mu(s)\right)\right] ds.
\end{aligned}$$

By Lemma 3.6, one can verify that $\{\xi_k(\cdot)\}$ and $\{\zeta_k(\cdot)\}$ are bounded sets in $L^q(0,T;E)$. It follows from Lemma 4.1 that we can assume that

$$\eta_k \longrightarrow \eta, \quad \omega_k \longrightarrow \omega, \qquad \text{in } C\left([0,\bar{t}], E_\alpha\right).$$

Due to reflexivity of E, $\{T^*(t);\, t \geq 0\}$ is a C_0-semigroup. For any $y^* \in E^*$,

$$\begin{aligned}
\left\langle y^*, \eta_k(t)\right\rangle_{E^*E} &= \int_0^t \left\langle y^*, T(t-\tau) G(\tau, y(\tau))\left(u_k(\tau)-\mu(\tau)\right)\right\rangle_{E^*E} d\tau \\
&= \int_0^T \left\langle \chi_{[0,t]}(\tau) G^*(\tau, y(\tau)) T^*(t-\tau) y^*, u_k(\tau)-\mu(\tau)\right\rangle_{XX^*} d\tau,
\end{aligned}$$

where $\chi_{[0,t]}$ is the characteristic function of $[0,t]$ and $\chi_{[0,t]}(\cdot)G^*(\cdot,y(\cdot))T^*(t-\cdot)y^* \in L^1(I,X)$. Since $u_k(\cdot) \longrightarrow \mu(\cdot)L^1(I,X)$ weakly in $L_w^\infty(I,X^*)$, we have $\langle y^*,\eta_k(t)\rangle_{E^*,E} \longrightarrow 0$. This implies that $\eta(\cdot)=0$ and $\eta_k \longrightarrow 0$ in $C([0,\bar{t}],E_\alpha)$. Hence,

$$\left\|\eta_k(\cdot)\right\|_{C([0,\bar{t}],E_\alpha)} \longrightarrow 0.$$

Similarly, one can verify that $\omega(\cdot)=0$ and $\omega_k \longrightarrow 0$ in $C([0,\bar{t}],E_\alpha)$. By virtue of Theorem 2.A, we have

$$\left\|y_k(t)-y(t)\right\|_\alpha \leq M\left(\left\|\eta_k(t)\right\|_\alpha+\left\|w_k(t)\right\|_\alpha\right).$$

The proof is completed. ∎

5. INTEGRODIFFERENTIAL INCLUSIONS

In this section, we discuss the equivalent of control integrodifferential equation and integrodifferential inclusion. We assume that E is reflexive and separable. U is a polish space.

Now we consider the following integrodifferential inclusion:

$$\begin{aligned} &\dot{y}(t) \in Ay(t) + P_t W(\cdot, y(\cdot), U), && t \in (0,T], \\ &y(t) = \varphi(t), && t \in [-r, 0], \end{aligned} \tag{5.1}$$

where $P_t W(\cdot, y(\cdot), U)$ is defined as follows: for $(s,y,u) \in [-r,T] \times E_\alpha \times U$, $W(s,y,u)$ is given by

$$W(s,y,u) = (g^*(s,y,u), h^*(s,y,u)) \in E \times E,$$

where

$$g^*(s,y,u) = \begin{cases} g(s,y,u), & s \in [0,T], \\ g(s,\varphi(s),u), & s \in [-r,0], \end{cases}$$

and

$$h^*(s,y,u) = \begin{cases} h(s,y,u), & s \in [0,T], \\ h(s,\varphi(s),u), & s \in [-r,0]. \end{cases}$$

For $y \in C(I_r, E_\alpha)$,

$$P_t W(\cdot, y(\cdot), U) = \left\{ g^*(t, y(t), u) + \int_{-r}^{t} \theta(t-s) h^*(s, y(s), u)\, ds \mid u \in U \right\}.$$

$y \in C(I_r, E_\alpha)$ is said to be a solution of (5.1), if there exists a pair of functions $(\tilde{g}, \tilde{h}) \in L^q(I,E) \times L^{\gamma_2}(I_r, E)$, $\tilde{g}(t) + \int_{-r}^{t} \theta(t-s)\tilde{h}(s)\,ds \in P_t W(\cdot, y(\cdot), U)$ a.e. and

$$\begin{aligned} y(t) &= T(t)\varphi(0) + \int_0^t T(t-\tau)\tilde{g}(\tau)\,d\tau \\ &\quad + \int_0^t T(t-\tau)\left[\int_{-r}^{\tau} \theta(\tau-s)\tilde{h}(s)\,ds\right] d\tau, && t \in (0,T], \\ y(t) &= \varphi(t), && t \in [-r,0], \end{aligned}$$

$(y(\cdot), \tilde{g}(\cdot), \tilde{h}(\cdot))$ is called a trajectory of (5.1).

We can prove the following equivalence theorem.

Theorem 5.A. *Under Assumptions [A], [G$_1$], and [H$_1$], $y(\cdot)$ satisfies (3.1) with $u(\cdot) \in U_{ad}$ if and only if it satisfies the differential inclusion (5.1).*

Proof. If $y(\cdot)$ is a solution of (3.1) with $u \in U_{ad}$. This means

$$\begin{aligned} y(t) &= T(t)\varphi(0) + \int_0^t T(t-\tau) g(\tau, y(\tau), u(\tau))\,d\tau \\ &\quad + \int_0^t T(t-\tau)\left[\int_{-r}^{\tau} \theta(\tau-s) h(s, y(s), u(s))\,ds\right] d\tau, && t \in (0,T], \\ y(t) &= \varphi(t), && t \in [-r,0], \quad u \in U_{ad}. \end{aligned}$$

Define $\tilde{g}(s) = g^*(s, y(s), u(s))$ and $\tilde{h}(s) = h^*(s, y(s), u(s))$. Obviously

$$\left(\tilde{g}(s), \tilde{h}(s)\right) = W(s, y(s), u(s)), \qquad \text{a.e. on } I_r.$$

By Assumptions [G$_1$], [H$_1$], $\tilde{g}(\cdot) \in L^q(I_r; E)$, $\tilde{h} \in L^{r_2}(I_r; E)$, $\tilde{g}(t) + \int_{-r}^{t} \theta(t-s)\tilde{h}(s)\,ds \in P_t W(\cdot, y(\cdot), U)$ a.e. on I_r. Hence, $(y(\cdot), \tilde{g}(\cdot), \tilde{h}(\cdot))$ is a trajectory of (5.1).

On the other hand, if $(y(\cdot), \tilde{g}(\cdot), \tilde{h}(\cdot))$ is a trajectory of (5.1), this means that $(\tilde{g}(s), \tilde{h}(s)) \in W(s, y(s), U)$ a.e. on I_r. U is polish space and $E \times E$ is a Banach space. For $y \in C([-r, T], E_\alpha)$ fixed

$$W(\cdot, y(\cdot), \cdot) : I_r \times U \longrightarrow E \times E.$$

By Assumptions [G$_1$] and [H$_1$], one can verify that:

(1) $t \longmapsto W(t, y(t), u)$ is strongly measurable in t for every $u \in U$ fixed.

(2) $u \longmapsto W(t, y(t), u)$ is locally uniformly continuous in U for every t fixed.

Thanks to Filippov theorem (see [6, Theorem 12.8.1]) there exists a strongly measurable U-valued function $u(\cdot)$ such that

$$W(s, y(s), u(s)) = \left(\tilde{g}(s), \tilde{h}(s)\right), \qquad \text{a.e. on } I_r.$$

That is,

$$\tilde{g}(t) = g(t, y(t), u(t)), \quad \tilde{h}(t) = h(t, y(t), u(t)), \qquad t \in I_r,$$

and

$$y(t) = T(t)\varphi(0) + \int_0^t T(t-\tau) g(\tau, y(\tau), u(\tau))\, d\tau + \int_0^t T(t-\tau) \int_{-r}^{\tau} \theta(\tau - s) h(s, y(s), u(s))\, ds\, d\tau.$$

This implies that $y(\cdot)$ is a solution of (3.1). ∎

Now we consider relaxed integrodifferential inclusion corresponding to relaxed system (3.3)

$$\begin{aligned} &\dot{y}(t) \in Ay(t) + \overline{\text{conv}} P_t W(\cdot, y(\cdot), U), && t \in (0, T], \\ &y(t) = \varphi(t), && t \in [-r, 0]. \end{aligned} \tag{5.2}$$

If there exists a pair of functions $(\tilde{g}, \tilde{h}) \in L^q(I_r, E) \times L^{r_2}(I_r, E)$ such that

$$\left(\tilde{g}(t) + \int_{-r}^{t} \theta(t-s) \tilde{h}(s)\, ds\right) \in \overline{\text{conv}} P_t W(\cdot, y(\cdot), U), \qquad \text{a.e. on } I_r,$$

and

$$\begin{aligned} y(t) &= T(t)\varphi(0) + \int_0^t T(t-\tau) \tilde{g}(\tau)\, d\tau \\ &\quad + \int_0^t T(t-\tau) \left[\int_{-r}^{\tau} \theta(\tau - s) \tilde{h}(s)\, ds\right] d\tau, && t \in (0, T], \\ y(t) &= \varphi(t), && t \in [-r, 0]. \end{aligned}$$

Then $y \in C(I_r, E_\alpha)$ is called a solution of (5.2) and $(y(\cdot), \tilde{g}(\cdot), \tilde{h}(\cdot))$ is said to be a trajectory of (5.2).

We also have an equivalence theorem.

THEOREM 5.B. *Let E be a reflexive separable Banach space, U is a normal topological space. Under Assumptions [A], [G$_1$], and [H$_1$], $y(\cdot)$ satisfies (3.3) with $\mu \in R(I, U)$ if and only if it satisfies the differential inclusion (5.2).*

PROOF. Suppose $y \in C(I_r, E_\alpha)$ is a solution of (3.3) for some $\mu \in R(I_r, U)$. Define $\tilde{g}(s) = G(s, y(s))\mu(s)$ and $\tilde{h}(s) = H(s, y(s))\mu(s)$, then $(\tilde{g}(\cdot), \tilde{h}(\cdot)) \in L^q(I, E) \times L^{r_2}(I_r, E)$. By [6, Lemma 14.4.3], one can verify that

$$\tilde{g}(s) \in \overline{\text{conv}} g(s, y(s), U), \qquad \text{a.e. and } \tilde{h}(s) \in \overline{\text{conv}} h(s, y(s), U), \quad \text{a.e. on } I_r.$$

This means $(\tilde{g}(s), \tilde{h}(s)) \in \overline{\text{conv}} W(s, y(s), U)$ a.e. on I_r, $(y(\cdot), \tilde{g}(\cdot), \tilde{h}(\cdot))$ is a trajectory of (5.2).

Conversely, $(y(\cdot), \tilde{g}(\cdot), \tilde{h}(\cdot))$ is a trajectory of (5.2). Set $\xi(\cdot) = (\tilde{g}(\cdot), \tilde{h}(\cdot))$, then $\xi(\cdot)$ is strongly measurable. By assumptions $\xi(t) \in \overline{\text{conv}} W(t, y(t), U) \equiv \overline{\text{conv}} \tilde{W}(t, U)$ a.e. on I_r, $\tilde{W}(t, \cdot) \in \text{BC}(U, E \times E)$ and for every $z^* \in E^* \times E^*$, $\langle z^*, \tilde{W}(t, \cdot)\rangle_{E^* \times E^*, E \times E} \in \text{BC}(U)$ in I_r and $t \longmapsto \langle z^*, \tilde{W}(t, \cdot)\rangle$ is a strongly measurable $\text{BC}(U)$-valued function. Note that $E \times E$ is a reflexive and separable Banach space and U is a normal topology space. By Lemma 14.4.4 of [6], there exists $\mu(\cdot) \in R(I_r, U)$ such that

$$\bar{W}(t)\mu(t) = \xi(t) \qquad \text{a.e., on } I_r,$$

where $\bar{W}(t)$ is defined by

$$\left\langle z^*, \bar{W}(t)\mu(t) \right\rangle = \int_U \left\langle z^*, \tilde{W}(t, u) \right\rangle_{E^* \times E^*, E \times E} \mu(t, du), \qquad \forall\, z^* \in E^* \times E^*.$$

Particularly let $z^* = (y_1^*, 0) \in E^* \times E^*$, then $\forall\, y_1^* \in E^*$,

$$\left\langle z^*, \bar{W}(t)\mu(t) \right\rangle = \int_U \langle y_1^*, g(t, y(t), u)\rangle_E\, \mu(t, du) = \langle y_1^*, G(t, y(t))\mu(t)\rangle .$$

At the same time, letting $z^* = (0, y_2^*)$, we have

$$\left\langle z^*, \bar{W}(t)\mu(t) \right\rangle = \langle y_2^*, H(t, y(t))\mu(t)\rangle .$$

Hence,

$$y(t) = T(t)y_0 + \int_0^t T(t-\tau)G(\tau, y(\tau))\mu(\tau)\, d\tau + \int_0^t T(t-\tau) \left[\int_0^\tau \theta(\tau - s)H(s, y(s))\mu(s)\, ds\right] d\tau.$$

This implies that $y(\cdot)$ is a solution of (3.3). ∎

6. EXISTENCE OF OPTIMAL CONTROLS

In this section, we discuss the existence of optimal control. We assume that E is a reflexive and separable Banach space.

We consider the original control problem (P) of minimizing the cost functional

$$J(y, u) = \int_0^T f_0(\tau, y(\tau), u(\tau))\, d\tau$$

among all trajectories of (3.1).

Suppose the following.

ASSUMPTION $[\text{F}_0]$. *$f_0 : I \times E_\alpha \times U \longrightarrow R$ satisfies*

(1) *$f_0(t, y, u)$ is bounded and continuous in u for (t, y) fixed and continuous in y for t fixed, uniformly respect to u.*

(2) *The function $t \longmapsto f_0(t, \cdot, \cdot)$ is a strongly measurable $\text{BC}(U)$-valued function and for every $e > 0$, there exists $K_0(\cdot, e) \in L^1(I)$ such that*

$$|f_0(t, y, u)| \le K_0(t, e)$$

provided $\|y\|_{E_\alpha} \le e$.

Define

$$F_0(t, y)\mu = \int_U f_0(t, y, \sigma)\mu\,(d\sigma), \qquad \mu \in \sum_{\text{rba}},$$

$$J_r(y, \mu) = \int_{I_r} F_0(t, y(t))\mu(t)\,(dt), \qquad \mu(\cdot) \in R\,(I_r, U)\,.$$

So-called relaxed control problem (P_r) is to minimize the cost functional $J_r(y, \mu)$ among all trajectories of (3.3).

LEMMA 6.1. *Under Assumptions [F$_0$], [H$_1$], and [G$_1$], F_0 is well defined and $t \longmapsto F_0(t, y(t))\mu(t)$ is strongly measurable for $\mu \in R(I_r, U)$, $y \in C(I_r, E_\alpha)$. Furthermore, if $\mu_k \longrightarrow \mu$ in $R(I_r, U)$ and $y_k \longrightarrow y$ in $C(I_r, E_\alpha)$ where y_k is the solution of (3.3) corresponding to μ_k, then y is the solution of (3.3) corresponding to μ and*

$$J_r(y, \mu) \leq \lim_{k\to\infty} J_r(y_k, \mu_k).$$

PROOF. By a similar argument of Lemma 3.6, $F_0(t, y)\mu$ is well defined for $\mu \in \sum_{\text{rba}}$. By virtue of dominated convergence theorem, using Assumption [F$_0$], one can verify that

$$f_0\left(\cdot, y_k\left(\cdot\right), \cdot\right) \longrightarrow f_0(\cdot, y(\cdot), \cdot), \qquad \text{in } L^1\left(I_r, \mathrm{BC}(U)\right).$$

According to the $L^1(I_r, \mathrm{BC}(U))$-weak convergence of sequence $\{\mu_k\}$ we have

$$J(y, \mu) = \lim_{k\to\infty} J\left(y_k, \mu_k\right).$$

Now we show that y is the solution of (3.3) corresponding to μ. Since y_k is a solution of (3.3) corresponding to μ_k, we have

$$y_k(t) = \omega(t) + \int_0^t T(t-\tau)G\left(\tau, y_k\tau)\right)\mu_k(\tau)\,d\tau + \int_0^t T(t-\tau)\int_0^\tau \theta(\tau - s)H\left(s, y_k(s)\right)\mu_k(s)\,ds\,d\tau,$$

then

$$\begin{aligned}\left\langle y_k\left(t\right), y^*\right\rangle_{EE^*} = \left\langle \omega\left(t\right), y^*\right\rangle_{EE^*} &+ \int_0^t \left\langle T\left(t-\tau\right)G\left(\tau, y_k\left(\tau\right)\right)\mu_k\left(\tau\right), y^*\right\rangle_{EE^*}\,d\tau \\ &+ \int_0^t \left[\int_0^\tau \left\langle T\left(t-\tau\right)\theta\left(\tau-s\right)H\left(s, y_k\left(s\right)\right)\mu_k\left(s\right), y^*\right\rangle_{EE^*}\,ds\right]d\tau.\end{aligned}$$

But

$$\int_0^t \left\langle T\left(t-\tau\right)G\left(\tau, y_k\left(\tau\right)\right)\mu_k\left(\tau\right), y^*\right\rangle_{EE^*}\,d\tau = \int_0^t \int_U \Pi_k\left(\tau, \sigma\right)\mu_k\left(\tau\right)\left(d\sigma\right)\,d\tau,$$

where

$$\Pi_k(\tau, \sigma) = \left\langle T(t-\tau)g\left(\tau, y_k(\tau), \sigma\right), y^*\right\rangle_{EE^*}$$

obviously $\sigma \longrightarrow \Pi_k(\tau, \sigma) \in \mathrm{BC}(U)$, $\Pi_k(\cdot, \cdot) \in L^1(I, \mathrm{BC}(U))$. By Assumption [G$_1$], we have

$$\Pi_k(\tau, \cdot) \longrightarrow \Pi(\tau, \cdot), \qquad \text{in } X = \mathrm{BC}(U), \quad \text{a.e. } \tau \in I.$$

There exists a constant $N > 0$ such that

$$|\Pi_k(\tau, \sigma)| \leq N = C_3(\tau) \in L^1(0, T).$$

Thanks to dominated convergence theorem, one can verify that

$$\Pi_k(\tau, \cdot) \longrightarrow \Pi(\tau, \cdot), \qquad \text{in } L^1(I, X),$$

and obtain that

$$\int_0^t \left\langle T(t-\tau)G\left(\tau, y_k(\tau)\right)\mu_k(\tau), y^*\right\rangle_{EE^*}\,d\tau \longrightarrow \int_0^t \left\langle T(t-\tau)G(\tau, y(\tau))\mu(\tau), y^*\right\rangle_{EE^*}\,d\tau.$$

Similarly, we can derive the same assertion for the last term. That is, for every $t \in I$, arbitrary $y^* \in E^*$, we have

$$\begin{aligned}\left\langle y^*, y(t)\right\rangle_{EE^*} = \left\langle y^*, \omega(t)\right\rangle &+ \int_0^t \left\langle y^*, T(t-\tau)G(\tau, y(\tau))\mu(\tau)\right\rangle\,d\tau \\ &+ \int_0^t \left[\int_0^\tau \left\langle y^*, T(t-\tau)\theta(\tau-s)H(s, y(s))\mu(s)\right\rangle\,ds\right]d\tau.\end{aligned}$$

This means that y is the solution of (3.3) corresponding to μ.

Similarly we have $J_r(y, \mu) \leq \lim_{k\to\infty} J_r(y_k, \mu_k)$. ∎

We can obtain the existence of relaxed optimal control for problem (P_r) immediately.

THEOREM 6.A. *Suppose [A_1], [F_0], [G_1], [H_1] hold. Then problem (P_r) has a relaxed optimal control $u \in R(I_r, U)$.*

PROOF. By Theorem 3.B, for every $\mu \in R(I_r, U)$ relaxed control system (3.3) has a unique mild solution $y \in C(I_r, E_\alpha)$. By Assumption [F_0] there exists $+\infty > M > 0$ such that

$$J(y,\mu) \geq -M, \qquad \forall\, \mu \in R(I_r, U).$$

We can find a minimizing sequence $\{\mu_k\}$ such that

$$\lim_{k\to\infty} J(y,\mu_k) = \inf_{\mu\in R(I,U)} J(y,\mu).$$

According to Lemma 3.3, we can assume $\mu_k \longrightarrow \mu$ in $R(I_r, U)$, i.e., $L^1(I,X)$ weakly in $L^\infty_\omega(I, X^*)$. Let $\{y_k(\cdot)\}$ be the corresponding solution sequence. It follows from assumptions on A, g, and h that $\{y_k(\cdot)\}$ is bounded in $C(I_r, E_\alpha)$. By Lemma 4.1 $\{y_k(\cdot)\}$ is precompact in $C(I_r, E_\alpha)$. We can assume

$$y_k(\cdot) \longrightarrow y(\cdot), \qquad \text{in } C(I_r, E_\alpha).$$

Using Lemma 6.1, one can verify the existence of relaxed optimal control. ∎

In order to get the existence result on original optimal control, we need generalized Cesari conditions.

Let U be a polish space and E a separable reflexive Banach space. Suppose $\varphi_1, \varphi_2 : I_r \times U \longrightarrow E$, $\varphi_0 : I_r \times U \longrightarrow R$ such that

(a) $u \longmapsto \varphi_1(t,u)$, $u \longmapsto \varphi_2(t,u)$, $u \longmapsto \varphi_0(t,u)$ are bounded and locally uniformly continuous in u for each t fixed;
(b) $t \longmapsto \langle y^*, \varphi_1(t,\cdot)\rangle$, $t \longmapsto \langle y^*, \varphi_2(t,\cdot)\rangle$ are strongly measurable BC(U)-valued functions for every $y^* \in E^*$;
(c) for every t, the set

$$\varepsilon\,(t;\varphi_0,\varphi_1,\varphi_2) = \{(a,e_1,e_2) \in R\times E\times E : a \geq \varphi_0(t,u),\ e_1 = \varphi_1(t,u),\ e_2 = \varphi_2(t,u),\ u\in U\}$$

is convex and closed. Then we have following lemma.

LEMMA 6.2. *Let $\mu \in R(I,U)$ be given. There exists a strongly measurable U-valued function $u(\cdot)$ such that*

$$\varphi_0(t,u(t)) \leq \int_U \varphi_0(t,u)\mu(t)\,(du), \qquad \text{a.e. on } I,$$
$$\varphi_1(t,u(t)) = \int_U \varphi_1(t,u)\mu(t)\,(du), \qquad \text{a.e. on } I,$$
$$\varphi_2(t,u(t)) = \int_U \varphi_2(t,u)\mu(t)\,(du), \qquad \text{a.e. on } I.$$

PROOF. Define $\omega : I_r \times (R_+ \times U) \longrightarrow R\times E\times E$ as follows:

$$\omega(t,(a,u)) = (a + \varphi_0(t,u), \varphi_1(t,u), \varphi_2(t,u))\,.$$

This function is locally uniformly continuous in $(R_+ \times U)$ for t fixed and strongly measurable in t

$$\omega(t, R_+\times U) = \varepsilon(t;\varphi_0,\varphi_1,\varphi_2).$$

Given $\mu \in R(I,U)$, define a function $\bar z(\cdot)$ with value in $R\times E\times E$ by

$$\bar z(t) = (\Phi_0(t)\mu(t), \Phi_1(t)\mu(t), \Phi_2(t)\mu(t)),$$

where

$$\langle y^*, \Phi_1(t)\mu(t)\rangle = \int_U \langle y^*, \varphi_1(t,u)\rangle\, \mu(t)\,(du),$$
$$\langle y^*, \Phi_2(t)\mu(t)\rangle = \int_U \langle y^*, \varphi_2(t,u)\rangle\, \mu(t)\,(du),$$
$$\Phi_0(t)\mu(t) = \int_U \varphi_0(t,u)\mu(t)\,(du),$$

for every $y^* \in E^*$. According to Lemma 14.4.4 of [6] we can assert that $\bar{z}(\cdot)$ is strongly measurable. Obversely, $(\bar{R}_+ \times U)$ is a polish space and $R \times E \times E$ is a Banach space. Lemma 14.4.3 of [6] implies that

$$\bar{z}(t) \in \overline{\mathrm{conv}}\,\omega\,(t; R_+ \times U) = \overline{\mathrm{conv}}\,\varepsilon(t;\varphi_0,\varphi_1,\varphi_2).$$

Due to convexity and closedness of ε, we have

$$\bar{z}(t) \in \varepsilon(t;\varphi_0,\varphi_1,\varphi_2).$$

By virtue of the implicit function theorem (see 12.8.2 of [6]) there exists a strongly measurable $R_+ \times U$-valued function $(a_0(\cdot), u(\cdot))$ such that

$$\bar{z}(t) = \omega(t;(a_0(t), u(t))),$$

that is,

$$a_0(t) + \varphi_0(t,u(t)) = \Phi_0(t)\mu(t), \qquad \varphi_1(t,u(t)) = \Phi_1(t)\mu(t), \qquad \varphi_2(t,u(t)) = \Phi_2(t)\mu(t). \quad \blacksquare$$

Now we can show the existence of original optimal controls. In addition to assumption of Theorem 6.A, we assume that $(t,y,u) \longmapsto g(t,y,u)$, $h(t,y,u)$, $f_0(t,y,u)$ are locally uniformly continuous in u for t, y fixed and the set

$$\varepsilon\,(t,y;f_0,g,h) = \{(a,e_1,e_2) \in R \times E \times E : a \geq f_0(t,y,u),\\ e_1 = g(t,y,u),\ e_2 = h(t,y,u),\ u \in U\},$$

for every $(t,y) \in I \times E_\alpha$, E is convex and closed (this is called generalized Cesari condition). By Theorem 6.A there is an optimal trajectory $(y(\cdot),\mu(\cdot))$ of (P_r). It follows from assumptions and Lemma 6.2 that there is a strongly measurable U-valued function such that

$$f_0(t,y(t),u(t)) \leq F_0(t,y(t))\mu(t), \qquad \text{a.e. on } I,$$
$$g(t,y(t),u(t)) = G(t,y(t))\mu(t), \qquad \text{a.e. on } I,$$
$$h(t,y(t),u(t)) = H(t,y(t))\mu(t), \qquad \text{a.e. on } I.$$

We can assert $(y(\cdot),u(\cdot))$ is an optimal trajectory of (P). That is, the following theorem holds.

THEOREM 6.B. *Under assumptions stated above, original control problem (P) has an original optimal control.*

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